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## STRUCTURE PRESERVING SPATIAL DISCRETIZATION OF A 1-D PIEZOELECTRIC TIMOSHENKO BEAM\*

T. VOSS<sup>†</sup> AND J. M. A. SCHERPEN<sup>‡</sup>

**Abstract.** In this paper we show how to spatially discretize a distributed model of a piezoelectric beam representing the dynamics of an inflatable space reflector in port-Hamiltonian (pH) form. This model can then be used to design a controller for the shape of the inflatable structure. Inflatable structures have very nice properties, suitable for aerospace applications, e.g., inflatable space reflectors. With this technology we can build inflatable reflectors which are about 100 times bigger than solid ones. But to be useful for telescopes we have to achieve the desired surface accuracy by actively controlling the surface of the inflatable. The starting point of the control design is modeling for control. In this paper we choose lumped pH modeling since these models offer a clear structure for control design. To be able to design a finite dimensional controller for the infinite dimensional system we need a finite dimensional approximation of the infinite dimensional system which inherits all the structural properties of the infinite dimensional system, e.g., passivity. To achieve this goal first divide the one-dimensional (1-D) Timoshenko beam with piezoelectric actuation into several finite elements. Next we discretize the dynamics of the beam on the finite element in a structure preserving way. These finite elements are then interconnected in a physical motivated way. The interconnected system is then a finite dimensional approximation of the beam dynamics in the pH framework. Hence, it has inherited all the physical properties of the infinite dimensional system. To show the validity of the finite dimensional system we will present simulation results. In future work we will also focus on two-dimensional (2-D) models.

**Key words.** infinite dimensional port-Hamiltonian systems, structure preserving spatial discretization, piezoelectric beam, passivity

**AMS subject classifications.** 65M60, 93C20

**DOI.** 10.1137/100789038

**1. Introduction.** Inflatable structures are a very promising technology for space applications [7]. With this emerging technology one is able to build bigger space crafts, which are cheaper in terms of costs but still use the same space in the orbiting device. As a consequence, the developments may enable us to build bigger solar panels and reflectors.

Due to the fact that any inflatable structure is built out of a polymer casing which is folded on Earth and then inflated with a gas in space, an inflatable structure cannot have the same surface accuracy as a rigid body. This disadvantage is the reason why inflatable structures are currently not the best option for high accuracy situations. In order to eventually improve the surface accuracy by using piezoelectric elements, in this paper we focus on modeling for control of an inflatable space reflector; see Figure 1.1.

In order to change the shape of an inflatable structure, one could use smart materials which can change their properties on demand, e.g., piezoelectric polymers [15]. This means that with such materials it is actually possible to change the shape of an element by means of an applied voltage. Since these materials are made of polymers

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FIG. 1.1. An inflatable space reflector test setup of the company L'garde ([www.lgarde.com](http://www.lgarde.com)).

it is possible to build extremely thin actuators which can then be bonded to the casing of the inflatable structure. Moreover, to be able to change the shape of the reflecting surface locally, the actuators are spread out over the whole surface. If one applies a voltage to the actuators, the piezoelectric material will change its length. Also, due to the bonding to the shell of the reflector the reflecting surface will bend locally.

We show how to spatially discretize an infinite dimensional port-Hamiltonian (pH) model [19] of a nonlinear Timoshenko beam with piezo actuation. The reason why we treat a one-dimensional (1-D) Timoshenko beam and not directly a two-dimensional (2-D) model is that we first want to treat a simplification of the very complex plate model. Therefore, we consider a cut through the piezoelectric surface which we model as a nonlinear Timoshenko beam. This simpler problem, although still complex, will be the starting point for the design of a shape controller which is able to actively change the shape of the beam. The inside that we obtain for the 1-D case is then the starting point for discretization of the (more complex) 2-D case.

Note that the model we have derived in [19] is an infinite dimensional pH model in the framework. But since we would like to use control methods that are specifically developed for *finite dimensional* energy based models, such as interconnection and damping assignment [11], we need to spatially discretize the system. The spatially discretized model can serve as the basis for the design of a controller which is able to change the shape of the beam so that a desired shape is achieved. For the task of spatially discretizing an infinite dimensional pH model, one could use well known spatial discretization schemes, e.g., the finite element method [20]. However, these schemes have several disadvantages which make them less useful if one would like to design a controller for a specific task. First of all, the usual spatial discretization schemes, e.g., [20, 6, 3], assume that the boundary conditions are given, but to obtain a model useful for control one would like to use the boundaries to control the system. Secondly, other spatial discretization schemes destroy the pH structure and the passivity of the system and this destroys the structure we would like to use for control. The reason for preserving the pH structure is that the structure provides excellent opportunities to design a controller; i.e., the interconnection of a finite dimensional pH system with another pH system yields a closed-loop pH system, which possesses the profitable properties on passivity and stability that stem from a general pH structure. If we use this for control, we obtain in fact a rather simple control structure, which can be used for controlling nonlinear systems. Note that although we present some simulation results of the controlled system we do not treat the controller design. The details of control design are presented in [18]. We should mention that there exist spatial discretization schemes, e.g., [6, 3], that preserve the structure or energy of the system but for all these schemes the boundary conditions cannot be used as inputs to the system. Also they do not preserve the pH interconnection structure.

Therefore, we will apply a specialized spatial discretization scheme which enables us to use the boundary values as control inputs while preserving the structure of the system. The scheme was first described in [5]. Note that the authors of [5] treat only systems which are very small (2 states only) while we have systems with 8 states. This makes the spatial discretization more involving. Additionally, in this paper, we tackle a system which has a nonconstant Dirac structure while the paper mentioned above only treats constant Dirac structures. Treating a system with a nonconstant Dirac structure results in additional problems during the spatial discretization. In this paper we show how to overcome these problems. At last we also show how to incorporate infinite dimensional inputs into the system, where the original paper was only using boundary inputs. Furthermore, the system is a piezoelectric structure which is modeled in such a way that one defines the mechanical and electrical dynamics independently and then interconnects the two dynamics of the system by defining the energy exchange between the two domains; see [17]. This divide and conquer modeling approach is possible due to the excellent properties of the pH framework.

The paper is organized as follows. In section 2 we give a short overview of the concept of pH modeling and explain why this modeling framework is so suitable to design controllers. Next, in section 3, we introduce the model of a piezoelectric composite beam with a quasi-static electrical field in the pH framework. The infinite dimensional model that we propose in this paper is derived using a reasoning similar to [19]. However, we now treat the Timoshenko beam, while in [19] we only treated the Euler–Bernoulli beam. Next, in section 4 we show how to spatially discretize the infinite dimensional model of a piezoelectric beam with a quasi-static electrical field. The procedure is split in several parts. We start by defining boundary ports and

TABLE 1.1  
Used symbols.

Symbol	Meaning
$C^E$	Young's modulus
$e_{a/b}^{kB}$	boundary force related to velocity $k \in \{v_1, v_3, \dot{\theta}\}$
$e_{a/b}^l$	value of the approximated effort at $a/b$ , $l \in \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \varepsilon_1, \dots, \varepsilon_4, E\}$
$\mathbf{e}$	effort
$E$	electrical field
$f_{a/b}^{kB}$	boundary flow related to velocity $k \in \{v_1, v_3, \dot{\theta}\}$
$f_{ab}^l$	value of the approximated flow on $Z_{ab}$ , $l \in \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \varepsilon_1, \dots, \varepsilon_4, E\}$
$\mathbf{f}$	flow
$G^E$	shear modulus
$K_{\text{tot}}$	total kinetic energy
$P_{\text{tot}}$	total potential energy
$\mathbf{p}$	momenta
$q$	electrical charge distribution
$z$	spatial coordinate
$Z_{ab} = [a, b]$	interval for discretization
$Z = [0, L]$	spatial domain
$\varepsilon_{11}$	normal strain in the $x$ -direction
$\varepsilon_{13}$	shear strain
$\tilde{\varepsilon} = [u'_0, w', \phi, \phi']$	strain parameters
$\psi^l$	modified flow, $l \in \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \varepsilon_1, \dots, \varepsilon_4, E\}$
$\phi_e$	magnetic flux distribution
$\omega_{a/b}^l$	zero-form basis function, $l \in \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \varepsilon_1, \dots, \varepsilon_4, E\}$
$\omega_{ab}^l$	one-form basis function, $l \in \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \varepsilon_1, \dots, \varepsilon_4, E\}$

the spatially approximating basis functions. Then we have to discretize the infinite dimensional interconnection structure and redefine the total effort in the considered finite element. This enables us to define a finite dimensional interconnection structure. The last step is to derive a finite dimensional approximation to the stored energy. During this process we also focus on several related questions, such as including inputs, and show simulations results. Finally, in section 5 we conclude the paper with some recommendations for future research.

Note that the model that we propose can also be used for modeling other structures, namely any flexible structure with piezo actuation, e.g., for vibration control in civil engineering.

**2. Short introduction to pH system.** In this section we introduce the pH modeling framework; see [14, 2]. The reason why we use this framework to do modeling for control is that pH systems have specific properties which make them suitable for control design. The first property of pH systems is that they are port-based models. This means that in order to interconnect two or more pH systems, one simply connects the ports in a physical way. Hence, the interconnection of pH systems is quite natural. This property can also be exploited for large scale modeling. One divides then the modeling of a complex system in subparts and uses the interconnection properties of pH systems to obtain the full dynamics of the system. Furthermore, pH models give an energy representation of the dynamics. Hence, one can use energy based control schemes to design the controller; such an example is the interconnection and damping assignment (IDA-PBC) [11]. Finally, pH systems are automatically passive and this property is also very helpful when designing a suitable controller.

This framework was originally developed for the modeling of finite dimensional systems. However, the framework has been extended to the case of infinite dimensional systems; see, for example, [10, 9].

**2.1. Finite dimensional systems.** Here we recall briefly what finite dimensional pH systems are and refer the interested reader to [4]. A finite dimensional pH system in local coordinates can be described as

$$(2.1) \quad \begin{aligned} \dot{x} &= (J(x) - R(x)) \frac{\partial H}{\partial x}(x) + B(x)u \\ y &= B^\top(x) \frac{\partial H}{\partial x}(x), \end{aligned}$$

where

- $x = (x_1, \dots, x_n)$  expresses local coordinates in an  $n$ -dimensional state space manifold  $\mathcal{X} \subset \mathbb{R}^n$ ;
- $u \in \mathbb{R}^m$  and  $y \in \mathbb{R}^m$  are the inputs and outputs, respectively, and together they define the ports of the system;
- $J : \mathcal{X} \rightarrow \mathbb{R}^{n \times n}$  is the interconnection matrix and depends smoothly on  $x$ , and  $J(x)$  is skew-symmetric  $J(x) = -J^\top(x)$ ;
- $R : \mathcal{X} \rightarrow \mathbb{R}^{n \times n}$  is the resistance matrix and is symmetric positive semidefinite  $R(x) = R^\top(x) \geq 0$ , and  $R(x)$  depends smoothly on  $x$ ;
- $B : \mathcal{X} \rightarrow \mathbb{R}^{n \times m}$  is the input force matrix and depends smoothly on  $x$ ;
- $H(x) : \mathcal{X} \rightarrow \mathbb{R}$  with  $H(x) > c > -\infty \forall x \in \mathcal{X}$  the so called Hamiltonian of the system. Normally  $H(x)$  represents the stored energy in the system.

Note that for this system the energy-balancing property holds:

$$(2.2) \quad \begin{aligned} \frac{dH}{dt} &= \frac{\partial^\top H}{\partial x} \dot{x} = \frac{\partial^\top H}{\partial x} \left( (J(x) - R(x)) \frac{\partial H}{\partial x}(x) + B(x)u \right) \\ &= - \underbrace{\frac{\partial^\top H}{\partial x} R(x) \frac{\partial H}{\partial x}(x)}_{\geq 0} + \underbrace{\frac{\partial^\top H}{\partial x} B(x)}_{y^\top} u \leq y^\top u. \end{aligned}$$

So, the Hamiltonian is a storage function and, therefore, a candidate Lyapunov function for the unforced system. Also it follows from (2.2) that the system is passive.

The last property we would like to point out is that the interconnection of two finite dimensional pH systems is a finite dimensional pH system. This property can be exploited for finite dimensional control design which is based on shaping the energy system of the system to be controlled by interconnecting it with another passive system (the controller).

Later on we will often use the effort-flow form of finite dimensional pH system. Therefore, we now shortly introduce this notation. The system in the form (2.1) can be recast as

$$(2.3) \quad \begin{aligned} f &= (J(x) - R(x))e + B(x)u \\ y &= B^\top(x)e \end{aligned}$$

with flows  $f = \dot{x}$  and efforts  $e = \frac{\partial}{\partial x}H(x)$ .

Another concept which we will use is the energy flow, also called the net power of the system. The energy flow is defined as

$$P_{\text{net}} = e^\top f + y^\top u.$$

**2.2. Infinite dimensional systems.** An infinite dimensional pH system consists of a Hamiltonian that describes the stored energy and the interconnection structure, which interconnects the energy storing elements, much like a finite dimensional system. In [10, 9] damping is included, but for our considerations we neglect dissipation. The reason for this is that we consider a piezoelectric beam in complete vacuum and without any gravity. Hence, we do not have external damping influences. Moreover, the effect of internal damping of a polymer is so small that it has hardly any effect. Furthermore, inclusion of damping will increase the complicity of the derivation, but does not add additional insight. The energy function of an infinite dimensional system can be described as

$$H(x) = \int_V \mathcal{H}(x(z))dV,$$

where  $\mathcal{H}(x(z))$  is the energy density depending on the state  $x(z)$  at a specific point  $z \in V$  in the  $n$ -dimensional volume  $V \subseteq \mathcal{Z}$ . Here  $\mathcal{Z}$  describes the actual space where the volume  $V$  is located—for physical systems this is normally  $\mathbb{R}^3$  and is related to the positions in space. Note that in the physics literature  $x$  relates to the spatial coordinate, but in systems theory one uses  $x$  to denote the state of a system. In this paper we denote the spatial coordinate by  $z$  and the state of a system by  $x$ . For the sake of simplicity of notation, we neglect from now on the spatial dependency of our state  $x$  when there is no danger of confusion.

In the finite dimensional case we calculate the gradient of the Hamiltonian to define the equations of motions but this is not possible for the infinite dimensional case. Instead of calculating the gradient we have to calculate the variational derivative of  $H(x)$  which is defined as

$$\frac{\delta H}{\delta x} = \frac{\partial \mathcal{H}}{\partial x}(x).$$

We have to replace the interconnection matrix in the infinite dimensional setting with a formal skew-adjoint differential operator  $J(x(z)) = -J(x(z))^*$ . If we assume the operator  $J(x)$  can be stated as

$$(2.4) \quad J(x)\zeta(x) = \sum_{i=0}^N \left( P_i(x) \frac{\partial^i}{\partial x^i} \right) \zeta(x),$$

where  $\zeta(x)$  is an arbitrary function and the  $P_i(x)$  are  $n \times n$  matrices, then the formal adjoint  $J^*$  is given by

$$J(x)^* f(x) = \sum_{i=0}^N (-1)^i \frac{\partial^i}{\partial x^i} (P_i(x) f(x)).$$

Using these results we can now define an infinite dimensional pH system as

$$\dot{x}(z) = J(x(z)) \frac{\delta H}{\delta x}(x(z)).$$

Note that this system is autonomous. In order to add inputs and outputs we have two possibilities:

- Boundary ports

Here the input acts directly at the boundary of the spatial domain of our infinite dimensional systems and the outputs are functions of  $x_b = x|_{\partial V}$ . Then we have

$$\begin{aligned} u(t) &= B(x(z))x_b(t), \\ y(t) &= C(x(z))x_b(t), \end{aligned}$$

where  $B(x)$  and  $C(x)$  are boundary operators. An example for a boundary port is a force acting on one side of a flexible beam.

- Distributed ports

The second class of ports for infinite dimensional systems are the so called distributed ports. These ports influence the dynamics of the whole spatial domain of our system or only of a subdomain. An example of a distributed port is a pressure acting on a flexible structure. The dynamics of the system in this spatial domain where the actuation takes place can be described as

$$\begin{aligned} \dot{x}(z) &= J(x(z)) \frac{\delta H}{\delta x}(x(z)) + B(x(z))u(z), \\ y(z) &= B^\top(x(z)) \frac{\delta H}{\delta x}(x(z)), \end{aligned}$$

where the operator  $B$  is the input force operator and  $u$  is the given input.

As for the finite dimensional case (2.2), we can also prove that an infinite dimensional system in pH form is conserving energy

$$\begin{aligned}\frac{dH}{dt} &= \int_V \frac{\delta^\top H}{\delta x} \dot{x} dV = \int_V \frac{\delta^\top H}{\delta x} \left( J(x) \frac{\delta H}{\delta x} + B(x)u \right) dV \\ &= \int_V y^\top u dV.\end{aligned}$$

This system can also be written in the effort-flow form and is then given by

$$\begin{aligned}f &= J(x)e + B(x)u, \\ y &= B^\top(x)e,\end{aligned}$$

where

$$e = \frac{\delta H}{\delta x}, \quad f = \dot{x}.$$

The net power of an infinite dimensional system can be written as follows:

$$P_{\text{net}} = \int_V e^\top f + \int_{\partial V} u^\top y,$$

where the first term describes the flow of energy in the system and the second term describes the flow of energy into or from the system via the boundaries. We use the infinite dimensional pH framework introduced here to describe the dynamics of our infinite dimensional structure.

**2.3. Short introduction to the differential-geometric framework.** We now give a brief introduction to differential forms which we will use during the spatial discretization of the piezoelectric beam. Since we treat a 1-D spatial domain  $\mathcal{Z} \subset \mathbb{R}$ , we can distinguish between functions (zero-forms) and distributions (one-forms). A zero-form is the differential-geometric representation of a function. Hence, we can evaluate a zero-form at any point of the interval. Then it is clear that we use zero-forms to represent zero-dimensional physical effects, e.g., the stress (force) at a point of our spatial domain since we can measure the stress acting on a point.

A one-form is the differential-geometric representation of a distribution. It is obvious that we cannot evaluate a distribution at a specific point of the interval. To actually calculate a value we have to integrate over a subdomain of our spatial domain. So, we can use one-forms to represent 1-D physical effects. One example for a one-form in the beam is the strain of the beam. We cannot measure the strain of a point but we can measure the strain between two points. This means that the strain must be a one-form since one has to integrate over a strain distribution to calculate the actual change in length between two points.

Hence, if we consider the spatial coordinate  $z$  for our interval  $\mathcal{Z}$  then a zero-form is simply given by a function  $f(z) : \mathcal{Z} \rightarrow \mathbb{R}$ , while a one-form is given by  $g(z)dz$  for a certain density function  $g(z) : \mathcal{Z} \rightarrow \mathbb{R}$ . We denote the space of  $n$ -forms over the spatial domain  $\mathcal{Z}$  as  $\Omega^n(\mathcal{Z})$ . Next we will list some of the possible operators which are defined for  $n$ -forms.

One can transform a zero-form into a one-form by spatial differentiation  $\omega := \frac{\partial f}{\partial z} dz$ . In the coordinate-free language that we want to use, this is denoted as  $\omega = df$ , where “d” is called the exterior derivative, converting  $(n-1)$ -forms to



$n$ -forms. Another operator we introduce is the wedge product “ $\wedge$ ,” which given a  $k$ -form  $\omega_1$  and a  $l$ -form  $\omega_2$  creates a  $(k+l)$ -form. The last operator we use is the Hodge star operator  $*$  converting any  $k$ -form on the  $n$ -dimensional spatial domain  $\mathcal{Z}$  to a  $(n-k)$ -form. E.g., for our 1-D domain it holds that  $*\omega = g(z)$  and  $*f(z) = f(z)dz$ .

**3. Infinite dimensional model of the Timoshenko beam with quasi-static electrical field.** In this section we summarize the dynamics of a piezoelectric composite with a quasi-static electrical field in distributed pH form. We have chosen to treat a quasi-static electrical field due to the fact that this is mostly done if one treats piezoelectricity in engineering. For the complete derivation see [19]. We also treat a very thin layer of material which is in space; therefore we have neglected any dissipation effects.

We assume that the considered composite consists of a base layer (identified by the subscript “b”) to which a piezoelectric layer (subscript “p”) is bonded. The cross section of the beam is depicted in Figure 3.1. Moreover, and without loss of generality, we assume that the base layer has a constant thickness ( $2d_b$ ) and a constant height  $2h_b$  while its length is  $L$ . We define the origin of the  $z_2z_3$ -plane in the center of mass of the base layer. So, the cross sectional area of the base layer  $A_b$  is  $[-d_b, d_b] \times [-h_b, h_b]$ . The piezoelectric layer is bonded on top of the base layer. Let  $h_p$  denote the height of the piezoelectric layer and let the width of this layer be  $2d_p$ . Moreover, and without loss of generality, we assume that the width of this layer is symmetric with respect to the  $z_1$ -axis. Hence, the cross sectional area of the piezo layer  $A_p$  is  $[-d_p, d_p] \times [h_b, h_b + h_p]$ . To simplify the notation in the following paragraphs we define the total area  $A_{\text{tot}} = A_b + A_p$ , the first moment of area of the piezoelectric layer  $I_{0,p} = \int_{A_p} z_3 dA_p$ , and the second moment of area of the piezoelectric layer  $I_{1,p} = \int_{A_p} z_3^2 dA_p$ . Now that we have described the geometry of the composite we define an expression of the strain in the beam.

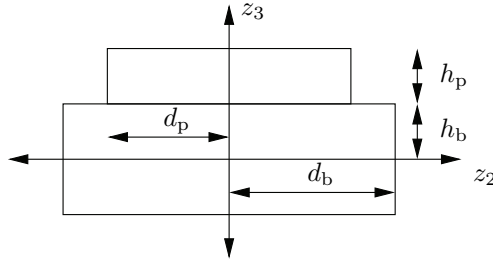


FIG. 3.1. Cross sectional area of the composite.

The standard assumption for a Timoshenko beam (see [12, 13]) is that there exist only two strains in the beam. All other strains are considered to be zero. The first one  $\varepsilon_{11}$  is the normal strain in  $z_1$  direction while the second strain  $\varepsilon_{13}$  describes the shear strain in the direction  $z_1z_3$ . Let  $f' = \frac{\partial}{\partial z_1}f$  denote the prime operator. Then these strains are derived as follows:

$$\begin{aligned}\varepsilon_{11} &= u'_0 - z_3\phi' + \frac{1}{2}(w')^2 + \frac{1}{2}(u'_0 - z_3\phi')^2, \\ \varepsilon_{13} &= \frac{1}{2}(w' - \phi) - \frac{1}{2}(u'_0 - z_3\phi')\phi,\end{aligned}$$

where  $u_0$  is the deformation of the beam in direction  $z_1$ ,  $w$  is the deformation in direction  $z_3$ , and  $\phi$  is the rotation of the cross sectional area. Note that for a Timoshenko beam it holds that  $\phi \neq w'$  ( $\phi = w'$  yields the Euler–Bernoulli beam).

The total energy  $H_{\text{tot}}$  stored in the composite consists of the energy stored in the purely mechanical base layer  $H_b$  and the energy stored in the piezoelectric layer  $H_p$ . We can state  $H_{\text{tot}}$  as

$$(3.1) \quad H_{\text{tot}} = H_b + H_p = K_{\text{tot}} + P_{\text{tot}},$$

where  $K_{\text{tot}}$  is the total kinetic energy and  $P_{\text{tot}}$  is the total potential energy stored in the system. The kinetic and potential energies are given by

$$K_{\text{tot}} = \frac{1}{2} \int_0^L \mathbf{p}^\top \mathbf{M}_{\text{tot}}^{-1} \mathbf{p} dz_1,$$

$$P_{\text{tot}} = \frac{1}{2} \int_0^L \int_{A_{\text{tot}}} C_{\text{tot}}^E \varepsilon_{11}^2 + 2G_{\text{tot}}^E \varepsilon_{13}^2 dA_{\text{tot}} + \epsilon^e A_p E^2 dz_1,$$

where

$$\mathbf{p} = \mathbf{M}_{\text{tot}} \dot{\mathbf{u}}, \quad \mathbf{M}_{\text{tot}} = \mathbf{M}_b + \mathbf{M}_p,$$

$$\mathbf{u} = (u_0, w, \phi)^\top,$$

$$C_{\text{tot}}^E(z_3) = \begin{cases} C_b^E & \text{for all } z_3 \in [-h_b, h_b] \\ C_p^E & \text{for all } z_3 \in (h_b, h_b + h_p] \end{cases}$$

$$G_{\text{tot}}^E(z_3) = \begin{cases} G_b^E & \text{for all } z_3 \in [-h_b, h_b] \\ G_p^E & \text{for all } z_3 \in (h_b, h_b + h_p] \end{cases}$$

with  $M_{\text{tot}}$  the mass matrix of our systems,  $C_{\text{tot}}^E$  the Young's modulus,  $G_{\text{tot}}^E$  the shear modulus of the used materials, and  $E$  the applied electrical field. All these variables depend on the spatial coordinate  $z_1$  and on the time  $t$ . Note that the factor 2 stems from the fact that  $\varepsilon_{13} = \varepsilon_{31}$  so one always has 2 shear strains.

Let  $(\mathbf{p}, \tilde{\varepsilon}, E)^\top$  be the state variables of the infinite dimensional pH system. Note that we choose  $\tilde{\varepsilon}$  to contain the 4 strain parameters  $u'$ ,  $w'$ ,  $\phi$ , and  $\phi'$ . The reason for using 4 strain parameters is that the interconnection matrix  $J$  becomes constant. If we would have chosen only 3 strain parameters the interconnection matrix would become state dependent, and the spatial discretization scheme we want to use here is not able to deal with state dependent interconnection matrices. This explains our choice of strain parameters. Of course the models with 3 or 4 strain parameters have the same dynamics. Moreover, note that the states  $\phi$  and  $\phi'$  are interdependent.

Under this choice of state variables the gradient of the Hamiltonian with respect to the state variables is

$$\begin{bmatrix} \delta_{\mathbf{p}_1^{\text{tot}}} H \\ \delta_{\mathbf{p}_2^{\text{tot}}} H \\ \delta_{\mathbf{p}_3^{\text{tot}}} H \\ \delta_{u'_0} H \\ \delta_{w'} H \\ \delta_\phi H \\ \delta_{\phi'} H \\ \delta_E H \end{bmatrix} = \begin{bmatrix} \dot{u}_0 \\ \dot{w} \\ \dot{\phi} \\ \int_{A_{\text{tot}}} [C_{\text{tot}}^E \varepsilon_{11} (1 + u'_0 - z\phi) - 2G_{\text{tot}}^E \varepsilon_{13} \phi] dA_{\text{tot}} \\ \int_{A_{\text{tot}}} [C_{\text{tot}}^E \varepsilon_{11} (w') + G_{\text{tot}}^E \varepsilon_{13}] dA_{\text{tot}} \\ \int_{A_{\text{tot}}} [-G_{\text{tot}}^E \varepsilon_{13} (1 + u'_0 - z\phi)] dA_{\text{tot}} \\ \int_{A_{\text{tot}}} [-C_{\text{tot}}^E \varepsilon_{11} (z_3) (1 + u'_0 - z\phi) + 2G_{\text{tot}}^E \varepsilon_{13} (z_3 \phi)] dA_{\text{tot}} \\ \epsilon^e A_p E \end{bmatrix}.$$

The interconnection structure of the system which describes the energy flow in the system is given by

$$(3.2) \quad \begin{bmatrix} \dot{\mathbf{p}} \\ \dot{\varepsilon} \\ \dot{E} \end{bmatrix} = \begin{bmatrix} 0 & \tilde{J} \\ -\tilde{J}^* & 0 \end{bmatrix} \delta H + \mathbf{B} \begin{bmatrix} \mathbf{f}_u \\ \mathbf{f}_w \\ I_e \end{bmatrix},$$

$$\mathbf{y} = \mathbf{B}^\top \nabla H,$$

where

$$\tilde{J} = \begin{bmatrix} \frac{\partial}{\partial z_1} & 0 & 0 & 0 & g_1(x) \\ 0 & \frac{\partial}{\partial z_1} & 0 & 0 & g_2(x) \\ 0 & 0 & -1 & \frac{\partial}{\partial z_1} & g_3(x) \end{bmatrix},$$

$$g_1(x) = -\frac{e}{\epsilon^e A_p} \frac{\partial}{\partial z_1} ((A_p + A_p u'_0 - I_{0,p} \phi') \cdot x), \quad g_2(x) = -\frac{e}{\epsilon^e} \frac{\partial}{\partial z_1} (w' \cdot x),$$

$$g_3(x) = \frac{e}{\epsilon^e A_p} \frac{\partial}{\partial z_1} ((I_{0,p} + I_{0,p} u'_0 - I_{1,p} \phi') \cdot x),$$

$$\mathbf{B}^\top = \begin{bmatrix} 2d_b & 0 & -2d_b h_b & 0 & 0 & 0 & 0 \\ 0 & 2d_b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\epsilon^e A_e} \end{bmatrix}.$$

The constant  $\epsilon^e$  represents the permittivity of the piezoelectric material; the inputs  $\mathbf{f}_u$  and  $\mathbf{f}_w$  represent a distributed force in the  $u$  and  $w$  direction, respectively. The input  $I_e$  represents a current distribution on the electrodes. Note that the system can also be written in its differential-geometric form. Then the interconnection matrix, efforts, and flows are given by

$$\tilde{J} = \begin{bmatrix} d & 0 & 0 & 0 & g_1(x) \\ 0 & d & 0 & 0 & g_2(x) \\ 0 & 0 & -* & d & g_3(x) \end{bmatrix},$$

$$\mathbf{f} = [f^{p_i}, f^{\varepsilon_i}, f^E]^\top = \dot{x}, \quad \mathbf{e} = [e^{p_i}, e^{\varepsilon_i}, e^E]^\top = \delta H.$$

In the differential-geometric form,  $g_i$  (for  $i = 1, 2, 3$ ) is given by

$$g_1(x) = -\frac{e}{\epsilon^e A_p} d((A_p + A_p u'_0 - I_{0,p} \phi') \cdot x), \quad g_2(x) = -\frac{e}{\epsilon^e} d(w' \cdot x),$$

$$g_3(x) = \frac{e}{\epsilon^e A_p} d((I_{0,p} + I_{0,p} u'_0 - I_{1,p} \phi') \cdot x).$$

Moreover, also note that the interaction between the mechanical and electrical domain for this model is only in the interconnection structure. The variational derivative of the mechanical energy depends only on the mechanical states. The same remark holds for the electrical energy. This results from the fact that for now we have chosen to model a quasi-static electrical field which is a standard assumption in the field of piezoelectricity for engineers. The reason for this is that the magnetic field which will

emerge between the two electrodes is so small that it is most of the time omitted because it has hardly any influence on the dynamics.

Note that we here treat a system without any damping. This results from the fact that we treat a system in space, so in vacuum, which is made of a very thin membrane so that any viscous effects are so small that they can be neglected.

**4. Discretization of the 1-D Timoshenko beam with quasi-static electrical field.** Recall from section 1 that the purpose of the here derived finite dimensional model is to design a controller which actively changes the shape of our beam. Specifically we want to be able to use pH based finite dimensional control schemes such as IDA-PBC [11]. So, we have to first spatially discretize the infinite dimensional dynamics while preserving the pH structure. The method we apply here was first proposed in [5]. In this paper we extend the method from [5] in order to be able to treat systems which have a nonconstant interconnection structure and are multiphysics systems and therefore more complex.

Hence, the goal of this section is to derive a finite dimensional pH model of a piezoelectric beam by applying the scheme in [5]. In order to calculate a finite dimensional approximation to the dynamics of a beam with length  $L$ , we have to perform several steps. First we have to subdivide the interval  $Z = [0, L]$  into  $n$  subintervals. On each of these subintervals we discretize the dynamics of the system. Let  $Z_{ab} = [a, b]$  denote the subinterval on which we perform the discretization. On any other subinterval the same procedure will be performed. We proceed as follows: we start with the discretization of our interconnection structure and then we discretize the energy function of our system. This yields a finite dimensional model which approximates the dynamics of the given subdomain  $[a, b]$ . Note that we have to perform these steps for all  $n$  intervals. Then we interconnect the  $n$  finite dimensional pH models to achieve a lumped model. The interconnection of the  $n$  local lumped models approximates the dynamics of our infinite dimensional model. During this discretization approach the main goal is to preserve the pH structure of the system. To simplify the notation we use from now on  $\varepsilon$  instead of  $\tilde{\varepsilon}$  to denote our four strain states  $([u'_0, w', \phi, \phi'])$ . In the following section we refer to the  $i$ th element of  $\varepsilon$  by using  $\varepsilon_i$ .

We now consider the part of the piezoelectric composite on the interval  $Z_{ab} = [a, b]$ . Note that all the flows of our system are distributions (one-forms) and all efforts are functions (zero-forms). Also note that in order to simplify the calculations we neglect for the moment distributed ports. These will be added after the discretization. We will add the effect of an applied pressure to the beam after the spatial discretization.

The procedure will be divided in the following steps:

- approximate the efforts and flows;
- define the boundary over which the elements exchange energy;
- discretize the interconnection structure;
- define the finite dimensional efforts via the net power;
- formulate the finite dimensional interconnection structure;
- discretize the energy function.

All these steps combined yield then a finite dimensional approximation of the infinite dimensional dynamics of our piezoelectric composite on the interval  $Z_{ab}$ .

**4.1. Modified efforts and flows.** Before we start spatially discretizing the system we perform a transformation to our system in order to simplify the spatial discretization. Note that the interconnection matrix of the system (3.2) has two types of entries: the exterior derivative  $d$  and the Hodge star operator  $*$ . Most nonzero

entries in the interconnection matrix  $J$  depend on the exterior derivative  $d$ . These entries relate the one-form flow with the zero-form effort via spatial differentiation. The Hodge star operator  $*$  at  $\tilde{J}_{3,3}$  on the other side transforms the zero-form  $e_{\varepsilon_3}$  directly into a one-form. So, there is no need to approximate  $e^{\varepsilon_3}$  directly. Instead we can just combine  $e^{\varepsilon_3}$  with  $f^{p_3}$  and treat the modified system. Therefore, in order to combine the effort  $e^{\varepsilon_3}$  with  $f^{p_3}$  we use the following parametrization of the matrix  $\tilde{J}$ :

$$\begin{aligned}\tilde{J} &= \begin{bmatrix} d & 0 & 0 & 0 & g_1(x) \\ 0 & d & 0 & 0 & g_2(x) \\ 0 & 0 & -* & d & g_3(x) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -* & 0 & 0 \end{bmatrix} + \begin{bmatrix} d & 0 & 0 & 0 & g_1(x) \\ 0 & d & 0 & 0 & g_2(x) \\ 0 & 0 & 0 & d & g_3(x) \end{bmatrix} \\ &= \tilde{J}_0 + d\tilde{J}_1.\end{aligned}$$

We use this parametrization of the interconnection matrix to define the modified flow  $\psi$  as follows:

$$\psi := \mathbf{f} - \begin{bmatrix} 0 & \tilde{J}_0 \\ -\tilde{J}_0^* & 0 \end{bmatrix} \mathbf{e}.$$

Then we reformulate our equations of motion as presented next:

$$(4.1) \quad \psi = d \begin{bmatrix} 0 & \tilde{J}_1 \\ -\tilde{J}_1^* & 0 \end{bmatrix} \mathbf{e} + \mathbf{B}\mathbf{u}.$$

Since we have redefined only the flow ( $\mathbf{f} \rightarrow \psi$ ), it is clear that the model with the modified flows  $\psi$  will have the same dynamics as the original model (3.2). So, instead of using (3.2) to compute the first finite dimensional approximation we use (4.1).

**4.2. Approximation of efforts, flows, and boundary values.** Similar to the classical finite element approach we define basis functions  $\omega$  which are used to approximate the energy and coenergy variables in our system. In order to obtain the best approximation we first have to define the goals to be achieved with this approximation.

The main goal is to approximate the dynamics on the interval  $Z_{ab}$  in such a way that we achieve a finite dimensional system that is able to approximate the boundary behavior as well as possible. The reason for this is that pH modeling and control of pH systems depends on the exchange of energy via ports. Note that in the case of an infinite dimensional system on the interval  $Z_{ab}$  these ports relate to the behavior of system at the boundaries. This idea becomes clearer if one takes into account that we want to approximate the total dynamics on  $Z = [0, L]$  by an interconnection of  $n$  local systems which exchange energy via the boundary ports. Then it immediately follows that the approximation of the total dynamics is directly related to the approximation of the boundary values. Every approximation error that we make at the boundaries will propagate through the whole system. Also note that one would like to connect the system via the boundary ports to the actual controller. Here we will not do this because we are treating a distributed port and therefore we deal with a distributed input and not with a boundary input. Moreover, one could use the divide and conquer idea (splitting a complex modeling problem into several smaller modeling problems) to model a physical problem. Then it becomes clear that a good approximation of the boundary values is necessary to ensure a certain accuracy of the total model, because the interconnection of the submodels will be done via the boundary ports.

Another property which we would like to preserve as well as possible is the structure of the system. In particular we want the finite dimensional approximation of our

infinite dimensional pH system to also be a pH system. The reason for this is that we want to preserve the passivity and the structure of the system—these are very useful properties when one wants to design a controller or wants to analyze the behavior of a system.

All these considerations are playing an important role in the definition of the boundary ports and the approximation of the efforts and flows.

**4.2.1. Boundary ports.** We first define the flow of energy over the boundaries. From a physical perspective it is clear that we have 3 power ports at both sides of the spatial domain  $Z_{ab}$ . The reasoning behind this is as follows.

At each boundary of the spatial domain we have 3 different velocities, the ones in  $z_1$  and  $z_3$  direction (denoted by  $v_1$  and  $v_3$ , respectively), and the rotational velocity  $\dot{\theta}$ . Let  $F_1$ ,  $F_3$ , and  $\tau$  denote the forces corresponding to the directions in  $z_1$ ,  $z_3$ , and  $\theta$ , respectively. The product of the corresponding velocity and force pair defines then a power port, e.g.,  $(v_1, F_1)$ . But since velocities are flows we denote them for now as  $f_{a/b}^{kB}$ , where  $k \in \{v_1, v_3, \dot{\theta}\}$  is the corresponding velocity,  $a$  and  $b$  determine whether the flow is at the left or right side, and B shows that it is a boundary value. For the efforts at the boundaries we use a slightly different notation. The indicator for the left and right boundary stays unchanged (subscript  $a$  or  $b$ ), but to determine the direction of the force we use a related flow identifier; e.g., the force in the  $z_1$  direction at the left boundary is denoted as  $e_a^{v_1B}$ . This notation has the advantage that like this we are able to identify a specific power pair, e.g.,  $(e_a^{v_1B}, f_a^{v_1B})$ .

Note that there is no electrical power port since we model a quasi-static electrical field. So, we only have the effort part of the energy domain and we are unable to define a power exchange over the boundaries. A sketch of the port structure can be seen in Figure 4.1.

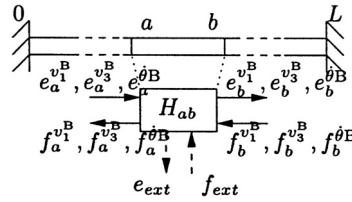


FIG. 4.1. Boundary ports of a finite element.

The next step is then to find a relation between the boundary ports and the efforts and flows of the system. As we already mentioned all power ports consist of a velocity and a force at the left or right boundary and, hence, they are zero-forms. This means that the boundary variables have to be related to the efforts of our system. From (3.2) it follows that the variational derivatives of the energy function with respect to momenta yield the velocities in the beam, while the variational derivatives with respect to  $[u'_0, w', \phi']$  yield the forces acting in the beam. This relation leads then to the following definition of the boundary ports.

1. Velocity/force in  $z_1$  direction

The first boundary power port consists of the velocity  $v_1$  and the force  $F_1$  in  $z_1$  direction. The relations between the boundary ports and the efforts are given by

$$(4.2) \quad \begin{aligned} F_1(\zeta) &= e^{\varepsilon_1}(\zeta) = e_{\zeta}^{v_1B}, \\ v_1(\zeta) &= e^{p_1}(\zeta) = f_{\zeta}^{v_1B}, \end{aligned}$$

where  $\zeta \in \{a, b\}$ .

## 2. Velocity/force in $z_3$ direction

The second boundary port consists of the velocity  $v_3$  and the force  $F_3$  in  $z_3$  direction. The relations between the boundary ports and the efforts are given by

$$(4.3) \quad \begin{aligned} F_3(\zeta) &= e^{\varepsilon_2}(\zeta) = e_{\zeta}^{v_3 B}, \\ v_3(\zeta) &= e^{p_2}(\zeta) = f_{\zeta}^{v_3 B}. \end{aligned}$$

## 3. Torque/angular velocity

The last port consists of the torque  $\tau$  and the angular velocities  $\dot{\theta}$  at the boundaries and is given by

$$(4.4) \quad \begin{aligned} \tau_{\zeta} &= e^{\varepsilon_3}(\zeta) = e_{\zeta}^{\dot{\theta} B}, \\ \dot{\theta}_{\zeta} &= e^{p_3}(\zeta) = f_{\zeta}^{\dot{\theta} B}. \end{aligned}$$

**4.2.2. Approximation of efforts and flows.** The general idea used here to approximate the infinite dimensional efforts and flows with a finite dimensional value is that we split the spatial and temporal dependent efforts and flows into the product of two parts which are either spatial or temporal dependent. Then by integration over the spatial domain we eliminate the spatial dependency of the efforts and flows and have just the temporal part left which is then finite dimensional. For more details see [5].

First we define a way to approximate the efforts on the spatial domain  $Z_{ab}$ . As we have already discussed in section 4.2.1, the approximation of the two boundary values is one of our biggest concerns. It is also known that all the efforts are zero-forms; e.g., we can measure the velocity of any particle. So, we are going to approximate the efforts as follows:

$$e^l(z, t) \approx e_a^l(t)\omega_a^l(z_1) + e_b^l(t)\omega_b^l(z_1),$$

where  $l \in \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \varepsilon_1, \dots, \varepsilon_4, E\}$ . We choose  $e_{a/b}^l(t)$  and  $e_{a/b}^l(t)$  such that they approximate the physical value of the efforts. Additionally, we choose them such that they coincide with the boundary values; e.g.,  $e_a^{\mathbf{p}_1}(t) = f_a^{v_1 B}(t)$ . The zero-forms  $\omega_a^l(z_1)$  and  $\omega_b^l(z_1)$  are the basis functions which are used to approximate the original value spatially. Hence, these basis functions are only spatially dependent. Due to the fact that we have chosen  $e_a^l(t)$  and  $e_b^l(t)$  to be equal to the boundary values and that the boundary values are given by  $e^l(z_1, t)|_{\partial Z}$  it immediately follows that the approximating basis functions have to fulfill the following conditions:

$$\begin{aligned} \omega_a^l(a) &= \omega_b^l(b) = 1, \\ \omega_a^l(b) &= \omega_b^l(a) = 0. \end{aligned}$$

A possible choice of two basis functions can be found in Figure 4.2.

Next we want to define the approximation of the flows. Due to the fact that all flows are one-forms we have to choose as basis function also a one-form which is used for the spatial approximation. Similar to the efforts where we wanted a good approximation of the boundary values we chose here the basis functions such that we achieve a good approximation of the total flow of the spatial domain. Hence, we choose the following approximation:

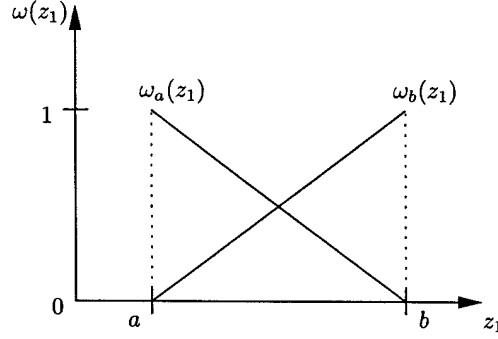


FIG. 4.2. Example for basis function of a 1-D finite element.

$$\psi^l(z_1, t) \approx \psi_{ab}^l(t) \omega_{ab}^l(z_1),$$

where  $\psi_{ab}^l(t)$  is the approximation of the total flow of the spatial domain. So, it holds that  $\psi_{ab}^l(t) \approx \int_{Z_{ab}} \psi^l(z_1, t)$ . And  $\omega_{ab}^l(z_1)$  is the basis function which spatially approximates the flow. Moreover, since the total flow is approximated by  $\psi_{ab}^l(t)$ , it immediately follows that the basis function  $\omega_{ab}^l(z_1)$  must fulfill the following condition:

$$\int_{Z_{ab}} \omega_{ab}^l = 1.$$

Given these two approximation schemes we can define the following approximation for all efforts and flows of our system:

Flows (one-forms)

$$(4.5) \quad \begin{aligned} \psi^{p_i}(z_1, t) &\approx \psi_{ab}^{p_i}(t) \omega_{ab}^{p_i}(z_1), \quad \psi^{\varepsilon_i}(z_1, t) \approx \psi_{ab}^{\varepsilon_i}(t) \omega_{ab}^{\varepsilon_i}(z_1), \\ \psi^E(z_1, t) &\approx \psi_{ab}^E(t) \omega_{ab}^E(z_1), \quad \int_{Z_{ab}} \omega_{ab} = 1. \end{aligned}$$

Efforts (zero-forms)

$$(4.6) \quad \begin{aligned} e^{p_i}(z_1, t) &= e_a^{p_i}(t) \omega_a^{p_i}(z_1) + e_b^{p_i}(t) \omega_b^{p_i}(z_1), \\ e^{\varepsilon_i}(z_1, t) &= e_a^{\varepsilon_i}(t) \omega_a^{\varepsilon_i}(z_1) + e_b^{\varepsilon_i}(t) \omega_b^{\varepsilon_i}(z_1), \\ e^E(z_1, t) &= e_a^E(t) \omega_a^E(z_1), \\ \omega_a(b) = \omega_b(a) &= 0, \quad \omega_a(a) = \omega_b(b) = 1. \end{aligned}$$

Due to the boundary value approximation goal we cannot choose any function as a basis function, because we have to ensure that the constraints are fulfilled. But these are not the only properties which the basis functions have to fulfill. We will see while deriving the final finite dimensional approximation that we have to obey more constraints to ensure the other approximation goal (to preserve the pH structure of the discretized system). In order to simplify the notation, in the following sections we will neglect the time and spatial dependency of all functions.

Due to the fact that the state  $x$  of the system is directly related to the flow  $f$ , ( $\dot{x} = f$ ), we can approximate the state using the same basis functions as for the flow

$$(4.7) \quad p_i = p_{ab}^i \omega_{ab}^{p_i}, \quad \varepsilon_i = \varepsilon_{ab}^i \omega_{ab}^{\varepsilon_i}, \quad E = E_{ab} \omega_{ab}^E.$$



Then we conclude that the following relation must hold:

$$\begin{bmatrix} f_{ab}^{p_i} \\ f_{ab}^{\varepsilon_i} \\ f_{ab}^E \end{bmatrix} = \frac{\partial}{\partial t} \begin{bmatrix} p_{ab}^i \\ \varepsilon_{ab}^i \\ E_{ab} \end{bmatrix}.$$

#### 4.2.3. Infinite dimensional approximation of the equations of motion.

Now we substitute the approximation of the flows and efforts defined above described by (4.5) and (4.6) into (4.1). This yields then an infinite dimensional approximation to the equations of motion; e.g., for the first equation of motion we get

$$(4.8) \quad \psi_{ab}^{p_1} \omega_{ab}^{p_1} = e_a^{\varepsilon_1} d\omega_a^{\varepsilon_1} + e_b^{\varepsilon_1} d\omega_b^{\varepsilon_1} + g_1(e_a^E \omega_a^E).$$

Note that these equations of motion still consist of flows which are one-forms and efforts which consist of zero-forms, but this approximation has a clear separation of the spatial and temporal coordinate. The separation between the spatial and temporal coordinates enables us to integrate over the spatial domain of interest  $Z_{ab}$ , which yields then a dynamical equation of motion with only a temporal coordinate.

**4.3. Discretization of the interconnection structure.** In this section we compute a finite dimensional version of the infinite dimensional approximation (4.8) of the last section. To do this we have to find relations between the one-forms and the zero-forms that we have chosen to approximate the infinite dimensional dynamics. We do this as proposed in [5].

From (4.8) it is clear that there must exist a relation between the one- and zero-forms such that (4.8) is fulfilled at all times. Therefore, we have to choose the one- and zero-forms in such a way that for every possible choice of  $e_a^l$ ,  $e_b^l$ , and  $\psi_{ab}^l$ , the approximated equations of motion are still fulfilled. Of course, this has to be done such that we preserve the pH structure of the system, since this is our main goal. One way to do this is to assume that every  $e_a^l$  and  $e_b^l$  is zero except one and calculate the relation between the one-form and remaining zero-form. We show how this can be done for (4.8).

Discretization procedure:

Assume that  $e_b^{\varepsilon_1}$  and  $e_a^E$  are zero. This is the case if there is electrical energy stored in the element and if the force  $e_b^{v_1 B}$  at the right hand side is equal to zero. As already discussed we have to choose the basis functions in such a way that even under these conditions the first equation of motion described by (4.8) still holds. We can then write the equation of motion as

$$\psi_{ab}^{p_1} \omega_{ab}^{p_1} = e_a^{\varepsilon_1} d\omega_a^{\varepsilon_1}.$$

If one now divides the equation above by  $e_a^{\varepsilon_1}$ , we can reformulate this equation as

$$c_1 \omega_{ab}^{p_1} = d\omega_a^{\varepsilon_1}$$

with  $c_1 = \psi_{ab}^{p_1} \cdot (e_a^{\varepsilon_1})^{-1}$ . Next, in order to calculate the value of  $c_1$  and with this relate  $\omega_{ab}^{p_1}$  with  $d\omega_a^{\varepsilon_1}$ , we integrate the equation above over  $Z_{ab}$ . This yields

$$c_1 \underbrace{\int_{Z_{ab}} \omega_{ab}^{p_1}}_{=1} = \underbrace{\omega_a^{\varepsilon_1}(b)}_{=0} - \underbrace{\omega_a^{\varepsilon_1}(a)}_{=1}.$$

Since we require that  $\omega_a^l(b) = 0$  and  $\int_{Z_{ab}} \omega_{ab}^l = \omega_a^l(a) = 1$  it follows that  $c_1$  has to be equal to  $-1$  in order to ensure the equation of motion. Hence, we find that the relation between  $\omega_{ab}^{p_1}$  and  $d\omega_a^{\varepsilon_1}$  can be described as follows:

$$-\omega_{ab}^{p_1} = d\omega_a^{\varepsilon_1}.$$

This means that the zero-form  $\omega_a^{\varepsilon_1}$  is completely determined up to a constant by the one-form  $\omega_{ab}^{p_1}$ .

In the same way we obtain that  $\omega_{ab}^{p_1} = d\omega_b^{\varepsilon_1}$ . Substituting this relation into (4.8) yields

$$\psi_{ab}^{p_1} \omega_{ab}^{p_1} = e_b^{\varepsilon_1} \omega_{ab}^{p_1} - e_a^{\varepsilon_1} \omega_{ab}^{p_1} - \frac{e}{\varepsilon^E} e_a^E \left( d\omega_a^E + u'_{ab} d(*\omega_{ab}^{\varepsilon_1} \wedge \omega_a^E) - \frac{I_{0,p}}{A_p} \phi'_{ab} d(*\omega_{ab}^{\varepsilon_4} \wedge \omega_a^E) \right).$$

If we then integrate the equation of motion over  $Z_{ab}$  and use that  $\int_{Z_{ab}} \omega_{ab}^{p_1} = 1$  we obtain a finite dimensional approximation to (4.8)

$$(4.9) \quad \psi_{ab}^{p_1} = (e_b^{\varepsilon_1} - e_a^{\varepsilon_1}) - \frac{e}{\varepsilon^E} e_a^E \left( \omega_a^E|_a^b + u'_{ab} (*\omega_{ab}^{\varepsilon_1} \wedge \omega_a^E)|_a^b - \frac{I_0}{A} \phi'_{ab} (*\omega_{ab}^{\varepsilon_4} \wedge \omega_a^E)|_a^b \right).$$

The above procedure has to be applied in the same way to all equations of motion. In order to derive the relations between the basis functions  $\omega_{ab}^j$ ,  $\omega_a^i$ , and  $\omega_b^i$  we again assume that all coefficients but one are equal to 0 and then we integrate over the interval  $Z_{ab}$ .

Note that (4.9) is a finite dimensional approximation for (4.1). So, we have to change the states from the modified flows to the original physical states that we have chosen. We show how this can be done for the third equation of motion, because the only nonzero row of  $\mathbf{J}_0$  is in the 3rd row. Since we have defined  $\psi = \mathbf{f} - \mathbf{J}_0 \mathbf{e}$  in the infinite dimensional case we can now insert the approximations for the efforts and flows, which yield

$$\varphi_{ab}^{p_3} \omega_{ab}^{p_3} = f_{ab}^{p_3} \omega_{ab}^{p_3} - (*e_a^{\varepsilon_3} \omega_a^{\varepsilon_3} + e_b^{\varepsilon_3} \omega_b^{\varepsilon_3}).$$

If we multiply this equation with the zero-forms  $*\omega_{ab}^{\varepsilon_3}$  and integrate over  $Z_{ab}$  we obtain the following relation between the modified flow  $\varphi_{ab}^{p_3}$  and  $f_{ab}^{p_3}$ :

$$\varphi_{ab}^{p_3} = f_{ab}^{p_3} + \frac{1}{\int_{Z_{ab}} \omega_{ab}^{p_3} \wedge *\omega_{ab}^{\varepsilon_3}} e_a^{\varepsilon_3} \int_{Z_{ab}} \omega_a^{\varepsilon_3} \omega_{ab}^{\varepsilon_3} - \frac{1}{\int_{Z_{ab}} \omega_{ab}^{p_3} \wedge *\omega_{ab}^{\varepsilon_3}} e_b^{\varepsilon_3} \int_{Z_{ab}} \omega_b^{\varepsilon_3} \omega_{ab}^{\varepsilon_3}.$$

Substituting the relation between the modified flows and the originals into the first finite dimensional approximation and assuming that the following conditions for the basis functions can be fulfilled

$$(4.10) \quad \begin{aligned} \omega_a^E|_a^b &= 1, \\ (*\omega_{ab}^{\varepsilon_1} \wedge \omega_a^E)|_a^b &= \int_{Z_{ab}} *\omega_{ab}^{\varepsilon_1} \wedge \omega_a^{\varepsilon_1} = \int_{Z_{ab}} *\omega_{ab}^{\varepsilon_1} \wedge \omega_{ab}^{\varepsilon_4}, \\ (*\omega_{ab}^{\varepsilon_4} \wedge \omega_a^E)|_a^b &= \int_{Z_{ab}} *\omega_{ab}^{\varepsilon_4} \wedge \omega_a^{\varepsilon_1} = \int_{Z_{ab}} *\omega_{ab}^{\varepsilon_4} \wedge \omega_{ab}^{\varepsilon_4}, \\ (*\omega_{ab}^{\varepsilon_2} \wedge \omega_a^E)|_a^b &= \int_{Z_{ab}} *\omega_{ab}^{\varepsilon_2} \wedge \omega_a^{\varepsilon_2} \end{aligned}$$

yields the following finite dimensional approximation of our system:

$$\begin{aligned}
 (4.11) \quad f_{ab}^{p_1} &= e_b^{\varepsilon_1} - e_a^{\varepsilon_1} + c_1(u'_{ab}, \phi'_{ab})e_a^E, \\
 f_{ab}^{p_2} &= e_b^{\varepsilon_2} - e_a^{\varepsilon_2} + c_2(w'_{ab})e_a^E, \\
 f_{ab}^{p_3} &= -\frac{1}{\bar{\alpha}}(\alpha_{ab}^{\varepsilon_3}e_a^{\varepsilon_3} + \alpha_{ba}^{\varepsilon_3}e_b^{\varepsilon_3}) + e_b^{\varepsilon_4} - e_a^{\varepsilon_4} + c_3(u'_{ab}, \phi'_{ab})e_a^E, \\
 f_{ab}^{\varepsilon_1} &= e_b^{p_1} - e_a^{p_1}, \\
 f_{ab}^{\varepsilon_2} &= e_b^{p_2} - e_a^{p_2}, \\
 f_{ab}^{\varepsilon_3} &= \frac{1}{\bar{\alpha}}(\alpha_{ab}^{p_3}e_a^{p_3} + \alpha_{ba}^{p_3}e_b^{p_3}), \\
 f_{ab}^{\varepsilon_4} &= e_b^{p_3} - e_a^{p_3}, \\
 f_{ab}^E &= c_1(u'_{ab}, \phi'_{ab})(e_b^{p_1} - e_a^{p_1}) + c_2(w'_{ab})(e_b^{p_2} - e_a^{p_2}) + c_3(u'_{ab}, \phi'_{ab})(e_b^{p_3} - e_a^{p_3}),
 \end{aligned}$$

where

$$\begin{aligned}
 c_1(u'_{ab}, \phi'_{ab}) &= -\frac{e}{\epsilon^e} \left( 1 + u'_{ab} \int_{Z_{ab}} * \omega_{ab}^{\varepsilon_1} \wedge \omega_{ab}^{\varepsilon_1} - \frac{I_0}{A} \phi'_{ab} \int_{Z_{ab}} * \omega_{ab}^{\varepsilon_4} \wedge \omega_{ab}^{\varepsilon_1} \right), \\
 c_2(w'_{ab}) &= -\frac{e}{\epsilon^e} w'_{ab} \int_{Z_{ab}} * \omega_{ab}^{\varepsilon_2} \wedge \omega_{ab}^{\varepsilon_2}, \\
 c_3(u'_{ab}, \phi'_{ab}) &= \frac{eI_0}{\epsilon^e A} \left( 1 + u'_{ab} \int_{Z_{ab}} * \omega_{ab}^{\varepsilon_1} \wedge \omega_{ab}^{\varepsilon_1} - \frac{I}{I_0} \phi'_{ab} \int_{Z_{ab}} * \omega_{ab}^{\varepsilon_4} \wedge \omega_{ab}^{\varepsilon_1} \right), \\
 \bar{\alpha} &= \frac{1}{\int_{Z_{ab}} \omega_{ab}^{\varepsilon_3} \wedge * \omega_{ab}^{p_3}}, \quad \alpha_{ab}^l = \int_{Z_{ab}} \omega_a^l \omega_{ab}^l, \quad \alpha_{ba}^l = \int_{Z_{ab}} \omega_b^l \omega_{ab}^l.
 \end{aligned}$$

The additional conditions (4.10) ensure that the finite dimensional approximation is still skew-symmetric, so we can summarize that treating nonconstant interconnection structures usually results in more constraints on the basis functions. Of course it could happen that we are not able to find basis functions which are fulfilling the derived constraints. But this is here not the case; e.g., the following basis functions fulfill all derived constraints:

$$\begin{aligned}
 \omega_a^l &= \frac{b - z_3}{b - a}, & \omega_b^l &= \frac{z_3 - a}{b - a}, \\
 \omega_{ab}^l &= \frac{1}{b - a}, & \omega_a^E &= \frac{z_3 - a}{b - a}, \\
 \omega_{ab}^E &= \frac{1}{b - a}.
 \end{aligned}$$

This system is not yet in pH form; see section 2—we have not yet defined the interconnection structure, also the equations of motion depend right now only on the boundary values  $e_{a/b}^l$  and not on an effort which is defined in  $Z_{ab}$ . So, in order to be able to derive the equations of motions in pH form we will have to define the total effort of  $Z_{ab}$  (see section 4.4).

**4.4. Definition of the total effort via the net power of the system.** As we can see in (2.3) the effort of a finite dimensional pH system is a vector. But up to now we have expressed the effort in the subdomain  $Z_{ab}$  by two boundary values  $e_a^l$  and  $e_b^l$ . So, in order to define the finite dimensional interconnection structure we first have to define the total effort of the subdomain  $Z_{ab}$ . To this aim we will use the net power of the system in  $Z_{ab}$ .

The infinite dimensional net power of the system on the interval  $Z_{ab}$  is given by

$$(4.12) \quad P_{ab}^{\text{net}} = \sum_{i=1}^3 \int_{Z_{ab}} e^{p_i} f^{p_i} + \sum_{i=1}^4 \int_{Z_{ab}} e^{\varepsilon_i} f^{\varepsilon_i} + \int_{Z_{ab}} e^E f^E.$$

Note that for ease of presentation we have neglected for the moment the boundary ports. If we insert our approximation for the flows and efforts described by (4.5) and (4.6) in (4.12) we can formulate a finite dimensional approximation of the net power as follows:

$$P_{ab}^{\text{net}} = \sum_{i=1}^3 e_{ab}^{p_i} f_{ab}^{p_i} + \sum_{i=1}^4 e_{ab}^{\varepsilon_i} f_{ab}^{\varepsilon_i} + e_{ab}^E f_{ab}^E,$$

where we used the following notation:

$$\begin{aligned} e_{ab}^l &= \alpha_a^l e_a^l + \alpha_b^l e_b^l, \\ \alpha_{ab}^l &= \int_{Z_{ab}} \omega_a^l \omega_{ab}^l, \quad \alpha_{ba}^l = \int_{Z_{ab}} \omega_b^l \omega_{ab}^l, \\ e_{ab}^E &= \alpha^E e_a^E, \quad \alpha^E = \int_{Z_{ab}} \omega_a^E \omega_{ab}^E \end{aligned}$$

with  $e_{ab}^*$  the approximation of the total effort in the system ( $e_{ab}^*$  depends on the boundary values of our spatial domain). This definition of the total effort on  $Z_{ab}$  can now be used to define the interconnection structure of our system.

**4.5. Formulation of the interconnection structure.** To be able to derive the interconnection structure for our finite dimensional pH system we define the following states, inputs, and outputs:

$$(4.13) \quad \begin{aligned} \dot{\mathbf{x}}_{ab} &= \mathbf{f}_{ab}, \quad \nabla_{\mathbf{x}_{ab}} H_{ab} = \mathbf{e}_{ab}, \\ \mathbf{u}_{ab} &= \left[ e_a^{v_1 B}, e_a^{v_3 B}, e_a^{\dot{\theta} B}, f_b^{v_1 B}, f_b^{v_3 B}, f_b^{\dot{\theta} B} \right]^\top, \\ \mathbf{y}_{ab} &= \left[ -f_a^{v_1 B}, -f_a^{v_3 B}, -f_a^{\dot{\theta} B}, e_b^{v_1 B}, e_b^{v_3 B}, e_b^{\dot{\theta} B} \right]^\top, \end{aligned}$$

where  $H_{ab}$  is the approximation of the Hamiltonian on  $Z_{ab}$  and will be defined in section 4.6. The choice of inputs and outputs is, as for any pH system, arbitrary but represents the most obvious choice. If one does not want to make the input-output choice yet, one can transform the system to the image-kernel representation; see [1, 5]. To simplify the notation we use the following properties of the basis functions:

$$(4.14) \quad 1 + \frac{\alpha_{ab}^l}{\alpha_{ba}^l} = \frac{1}{\alpha_{ba}^l}, \quad 1 + \frac{\alpha_{ba}^l}{\alpha_{ab}^l} = \frac{1}{\alpha_{ab}^l}, \quad \alpha_{ab}^{\varepsilon_i} = \alpha_{ba}^{p_i}, \quad \alpha_{ab}^{p_i} = \alpha_{ba}^{\varepsilon_i}.$$

Additionally we choose the basis functions such that  $\alpha^E = \alpha_{ab}^{p_i}$ .

Substitution of (4.14) and (4.13) into (4.11) yields then the pH system in input-output form

$$(4.15) \quad \begin{aligned} \dot{\mathbf{x}}_{ab} &= \mathbf{J}_{ab} \nabla_{\mathbf{x}_{ab}} H_{ab} + \mathbf{B}_{ab} \mathbf{u}_{ab}, \\ \mathbf{y}_{ab} &= \mathbf{B}_{ab}^\top \nabla_{\mathbf{x}_{ab}} H_{ab} + \mathbf{D}_{ab} \mathbf{u}_{ab}, \end{aligned}$$

where

$$\begin{aligned} \mathbf{J}_{ab} &= \begin{bmatrix} 0 & \mathbf{A} & C_1 \cdot (\alpha^E)^{-1} \\ -\mathbf{A}^\top & 0 & 0 \\ -C_1^\top \cdot (\alpha_{ab}^{p_1})^{-1} & 0 & 0 \end{bmatrix}, \quad \mathbf{B}_{ab} = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \\ 0 & (\alpha_{ab}^{p_1})^{-1} C_1 \end{bmatrix}, \\ \mathbf{D}_{ab} &= \begin{bmatrix} 0 & -\mathbf{I}_3 \\ \mathbf{I}_3 & 0 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} (\alpha_{ba}^{\varepsilon_1})^{-1} & 0 & 0 & 0 \\ 0 & (\alpha_{ba}^{\varepsilon_2})^{-1} & 0 & 0 \\ 0 & 0 & -\frac{1}{\alpha} & (\alpha_{ba}^{\varepsilon_4})^{-1} \end{bmatrix}, \\ B_1 &= \text{diag}\left(\frac{1}{\alpha_{ba}^{\varepsilon_1}}, \frac{1}{\alpha_{ba}^{\varepsilon_2}}, \frac{1}{\alpha_{ba}^{\varepsilon_3}}\right), \quad B_2 = \begin{bmatrix} (\alpha_{ab}^{p_1})^{-1} & 0 & 0 \\ 0 & (\alpha_{ab}^{p_2})^{-1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (\alpha_{ab}^{p_3})^{-1} \end{bmatrix}, \\ C_1 &= [c_1, c_2, c_3]^\top. \end{aligned}$$

It is straight forward to see that this finite dimensional interconnection structure is skew-symmetric. Hence, we have found a finite dimensional approximation to the infinite dimensional interconnection structure. But for the system to be in the finite dimensional pH framework, we additionally have to discretize the energy function of our system.

**4.6. Approximation of the energy function.** In this section we discretize the energy function  $H$ . Recall that the Hamiltonian of our system is given by

$$H = \frac{1}{2} \int_0^L \mathbf{p}^\top \mathbf{M}^{-1} \mathbf{p} + \int_A C^E \varepsilon_{11}^2 + 2G^E \varepsilon_{13}^2 dA + \epsilon^e A E^2 dz_1.$$

If we reformulate  $\varepsilon_{11}$  and  $\varepsilon_{13}$  in the following way,

$$\varepsilon_{11}(\tilde{\varepsilon}) = h_{11}^\top(\tilde{\varepsilon})\tilde{\varepsilon}, \quad \varepsilon_{13}(\tilde{\varepsilon}) = h_{13}^\top(\tilde{\varepsilon})\tilde{\varepsilon},$$

we can reformulate the Hamiltonian as

$$H(\mathbf{p}, \tilde{\varepsilon}, E) = \frac{1}{2} \int_0^L \mathbf{p}^\top \mathbf{M}^{-1} \mathbf{p} + \tilde{\varepsilon}^\top \mathbf{C}(\tilde{\varepsilon}) \tilde{\varepsilon} + \epsilon^e A E^2 dz_1,$$

where

$$\mathbf{C}(\tilde{\varepsilon}) = \int_A C^E h_{11}(\tilde{\varepsilon}) h_{11}^\top(\tilde{\varepsilon}) + 2G^E h_{13}(\tilde{\varepsilon}) h_{13}^\top(\tilde{\varepsilon}) dA.$$

With the approximation of our state variables we can approximate the Hamiltonian on  $Z_{ab}$  as

$$(4.16) \quad H_{ab} = \frac{1}{2} \mathbf{p}_{ab}^\top \mathbf{M}_{ab}^{-1} \mathbf{p}_{ab} + \frac{1}{2} \tilde{\varepsilon}_{ab}^\top \mathbf{C}_{ab}(\tilde{\varepsilon}_{ab}) \tilde{\varepsilon}_{ab} + \frac{1}{2} \epsilon_{ab}^e E_{ab}^2,$$

where

$$\begin{aligned} \mathbf{p}_{ab} &= \begin{bmatrix} p_{ab}^1 \\ p_{ab}^2 \\ p_{ab}^3 \end{bmatrix}, \mathbf{M}_{ab}^{-1} = \int_{Z_{ab}} * \Omega_{ab}^p \wedge \mathbf{M}^{-1} \wedge \Omega_{ab}^p, \\ \tilde{\varepsilon}_{ab} &= \begin{bmatrix} u'_{ab} \\ w'_{ab} \\ \phi_{ab} \\ \phi'_{ab} \end{bmatrix}, \epsilon_{ab}^e = eA \int_{Z_{ab}} * w_{ab}^E \wedge w_{ab}^E, \\ \mathbf{C}_{ab}(\tilde{\varepsilon}_{ab}) &= \int_{Z_{ab}} * \Omega_{ab}^\varepsilon \wedge \mathbf{C}(\Omega_{ab}^\varepsilon \tilde{\varepsilon}_{ab}) \wedge \Omega_{ab}^\varepsilon, \\ \Omega_{ab}^p &= \text{diag}(\omega_{ab}^{p1}, \omega_{ab}^{p2}, \omega_{ab}^{p3}), \Omega_{ab}^\varepsilon = \text{diag}(\omega_{ab}^{\varepsilon1}, \omega_{ab}^{\varepsilon2}, \omega_{ab}^{\varepsilon3}, \omega_{ab}^{\varepsilon4}). \end{aligned}$$

This can then be used to define the values  $\mathbf{e}_{ab}$  since it holds that

$$e_{ab}^{p_i} = \frac{\partial H_{ab}}{\partial p_i}, e_{ab}^{\varepsilon_i} = \frac{\partial H_{ab}}{\partial \varepsilon_i}, e_{ab}^E = \frac{\partial H_{ab}}{\partial E}.$$

The combination of the discretized interconnection structure (4.15) and the approximated energy function (4.16) yields a pH system which approximates the dynamics of the infinite dimensional Timoshenko beam.

One problem still remains, namely that it can be shown that the system is not stabilizable; see [16]. Stabilizability in control means that one is able to stabilize the system around a desired equilibrium and it is a crucial property one has to ensure in order to be able to design a controller which performs the stabilization. The reason why the system is not stabilizable stems from the fact that two of the states ( $\phi$  and  $\phi'$ ) are dependent on each other. The problem can be overcome by deriving a coordinate change which specifically takes into account this property; for details see [16].

**4.7. Adding the external inputs.** As described in section 2.2 our system has two external inputs. The first one is the mechanical input, and relates to the pressure of the beam due to the inflation of the structure. This input can be considered as a disturbance because we are not able to control the pressure. The second one is the current which is introduced to the electrodes and then creates an electrical field to change the length of the piezoelectric elements of the structure. This is the control input we are able to use to achieve the desired shape. The system (4.15) does not have any of these inputs, because up to now we have treated an autonomous system. The only inputs are the ports which are needed to interconnect several of systems of the type (4.15) to achieve the global approximation to the infinite dimensional model. So, we are going to add the two external inputs again.

The external input matrix of the infinite dimensional model is given by

$$B = \begin{bmatrix} 2d_b & 0 & -2d_b h_b & 0 & 0 & 0 & 0 & 0 \\ 0 & 2d_b & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\epsilon^e A_e} \end{bmatrix}^\top$$

with the related inputs  $\mathbf{u} = [\mathbf{f}_u, \mathbf{f}_w, I_e]^\top$ . This defines the infinite dimensional input structure  $B\mathbf{u}$ . To obtain a finite dimensional expression of this input structure we simply integrate over  $Z_{ab}$  (this yields a finite dimensional expression of the external input matrix which we denote by  $\mathbf{B}_{ab}^{\text{ext}}$  while the input itself stays unchanged).

The input matrix  $\mathbf{B}_{ab}^{\text{ext}}$  is given by

$$\mathbf{B}_{ab}^{\text{ext}} = (b - a) \begin{bmatrix} 2d_b & 0 & -2d_b h_b & 0 & 0 & 0 & 0 & 0 \\ 0 & 2d_b & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\epsilon^e A_e} \end{bmatrix}^\top.$$

So, the finite dimensional model with external inputs can be written as

$$(4.17) \quad \begin{aligned} \dot{\mathbf{x}}_{ab} &= \begin{bmatrix} 0 & J \\ -J^\top & 0 \end{bmatrix} \nabla_{\mathbf{x}_{ab}} H_{ab} + \mathbf{B}_{ab}^{\text{int}} \mathbf{u}_{\text{int}} + \mathbf{B}_{ab}^{\text{ext}} \mathbf{u}_{\text{ext}}, \\ \mathbf{y}_{\text{int}} &= (\mathbf{B}_{ab}^{\text{int}})^\top \nabla_{\mathbf{x}_{ab}} H_{ab} + D_{ab} \mathbf{u}_{\text{int}}, \\ \mathbf{y}_{\text{ext}} &= (\mathbf{B}_{ab}^{\text{ext}})^\top \nabla_{\mathbf{x}_{ab}} H_{ab}, \end{aligned}$$

where  $\mathbf{B}_{ab}^{\text{int}}$  is the input matrix defined in (4.15),  $\mathbf{u}_{\text{int}}$  is the input defined in (4.15), and  $\mathbf{u}_{\text{ext}}$  is the external input. This model now approximates the total dynamics of (3.2) on the interval  $Z_{ab}$ .

**4.8. Interconnection of the subsystems and simulation results.** With the procedure from the past sections we can calculate  $n$  simple finite dimensional pH systems which describe the dynamics of the beam locally (on the interval  $[a_i, b_i]$ , where it holds that  $a_i = b_{i-1}$ ). In order to achieve a global model for the dynamics of our beam we have to interconnect the system in a simple manner.

In (4.13) we have defined the inputs and outputs of a local system. It is clear that the input for the  $i$ th system has to consist of efforts (forces) at the left boundary and the flows (velocities) of the right boundary. So, the  $i$ th system is interconnected to the  $(i-1)$ st and the  $(i+1)$ st system. More specifically it must be that  $e_a^{liB} = e_b^{l(i-1)B}$  and  $f_b^{liB} = f_a^{l(i+1)B}$ , where, e.g.,  $e_a^{liB}$  is the effort of the right system at the left boundary. This gives us an interconnection of the  $i$ th system with the neighboring systems, as illustrated in Figure 4.3.

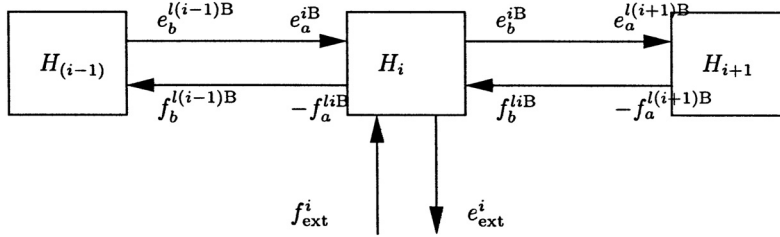
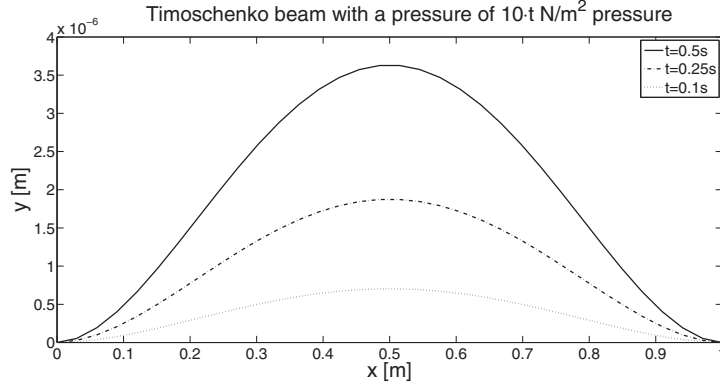
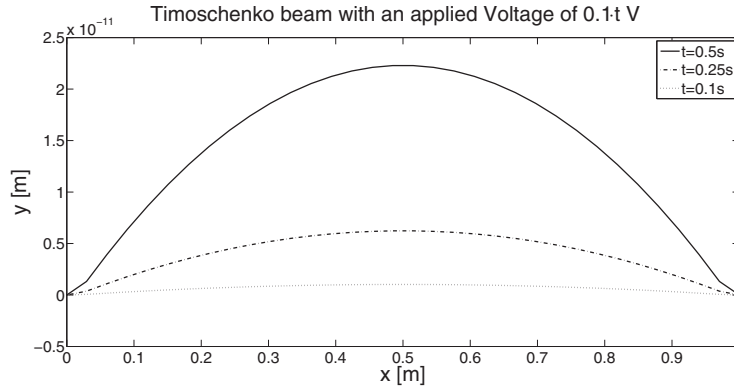


FIG. 4.3. Interconnection of the  $i$ -th system with the neighboring systems.

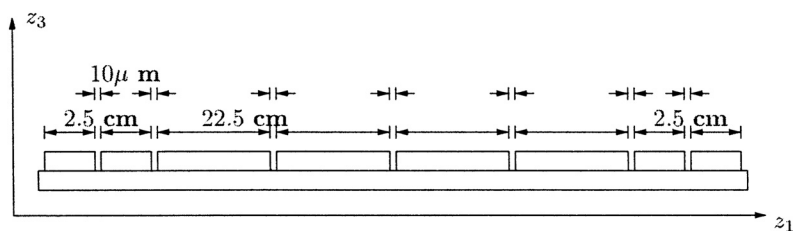
The following simulation in Matlab shows the behavior of a simple piezoelectric beam. We consider Kapton [8] as material for the base layer, and polyvinylidene fluoride (PVDF) [15] as piezoelectric material. The base layer has a length of 1 m, while the thickness and the width of the beam are 2 cm. The piezoelectric material covers the whole beam and has a thickness of 0.25 cm. For the first simulation we apply a pressure of  $10 \cdot t \frac{\text{N}}{\text{m}^2}$  until we reach a pressure of  $5 \frac{\text{N}}{\text{m}^2}$ . For the second simulation we apply a voltage of  $0.1 \cdot t \text{ V}$  until we reach a voltage of 0.5 V; see Figure 4.4 and 4.5. We show snapshots at time  $t \in \{0.1, 0.25, 0.5\}$ .

FIG. 4.4. *Simulation of a piezoelectric beam with pressure input.*FIG. 4.5. *Simulation of a piezoelectric beam with voltage input.*

Additionally we present some simulation results of a shape controlled beam. Therefore we consider a beam made of Kapton [8] with a length of 1 m, a height of  $25 \cdot 10^{-4}$  m, and a width of  $50 \cdot 10^{-4}$  m which is always clamped at the right sides. The clamping on the left side depends then on the shape we would like to achieve. Onto the beam we have bonded 8 piezoelectric patches of a thickness of  $50 \cdot 10^{-4}$  m and a height of  $25 \cdot 10^{-4}$  m. The distribution of the electrodes is the following. On the left and right side we have 2 piezoelectric patches with a length of  $25 \cdot 10^{-4}$  m. Each of the patches has a distance to the next patch of  $10^{-5}$  m. The rest of the 8 patches are distributed equally in the middle of the beam with a distance of  $10^{-5}$  m between each two consecutive patches. This structure is illustrated in Figure 4.6.

We apply an energy based control scheme (potential energy shaping), which is easy to implement for a system in the pH framework. The idea behind the control design can be sketched as following. In potential energy shaping one abuses this idea that injecting energy into the system will change the potential energy of the system. This idea can be exploited because a change of the potential energy of the system will also change the equilibrium of the system. Of course one injects the energy in such a way that the new equilibrium will then be the desired one. Additionally one can also inject energy with the effect of inducing damping into the system to render the

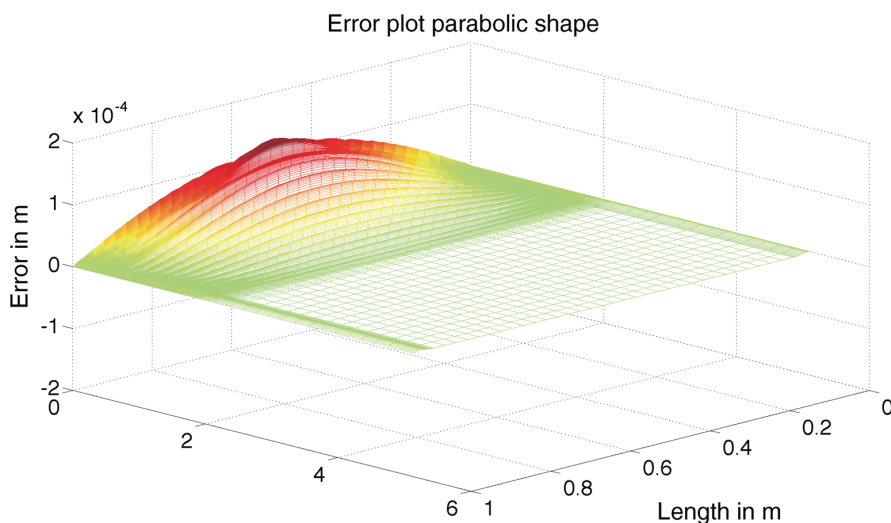


FIG. 4.6. *Beam structure.*

system asymptotically stable. Although the resulting controller is in fact a potential differential (PD) (linear) controller we can now proof asymptotical stability of our nonlinear closed-loop system. For more information about the control design we refer the reader to [18, 16]. We design the controller such that the following shapes will be achieved:

- Clamped left side. Achieve a parabolic shape described by  $z_3 = 4 \cdot 10^{-4}(-(z_1 - \frac{1}{2})^2 + \frac{1}{4})$ . This is a typical shape of a space reflector. For this shape it holds that  $w'(0/L) \neq 0$ .
- Unclamped left side. Achieve a linear shape described by  $z_3 = 10^{-4}(z_1 - 1)$ . This is a linear shape that we have chosen in order to show the potential of our controller in achieving shapes which do not fulfill the boundary conditions  $w'(L) = 0$ .

We can see the results for the parabolic shape in Figure 4.7. As we can see the controller reaches the optimal shape at the desired equilibrium after a short while. Additionally we see that at the boundary of the beam there is always a small error. This comes from the fact that we are trying to approximate a shape which violates the boundary condition of the beam. Therefore we will always have a small error at the boundaries for shapes which violate the boundary condition of the beam.

FIG. 4.7. *Error plot for a controller achieving a parabolic shape.*

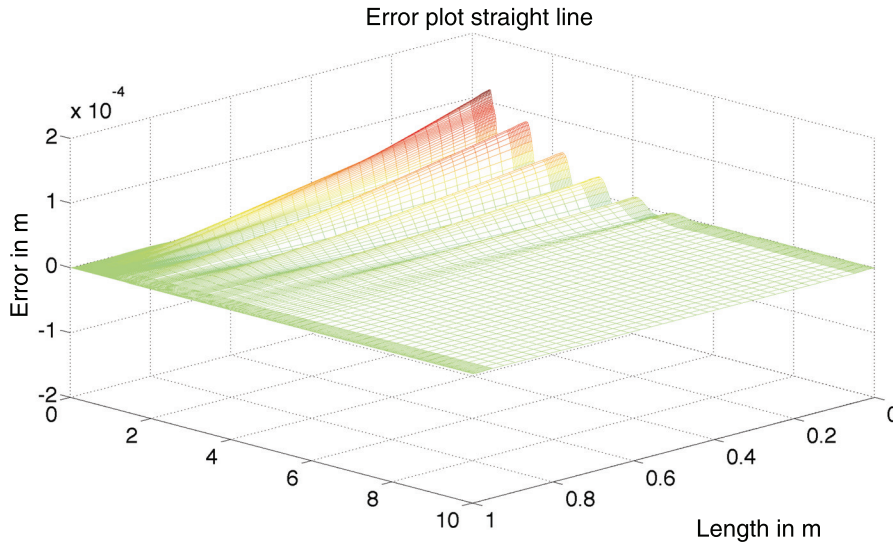


FIG. 4.8. Error plot for a controller achieving a straight line.

The results for the straight line are depicted in Figure 4.8. As we can see the controller stabilizes the shape of the beam around the desired equilibrium also for open beams. The error behavior is quite similar to the parabolic case. But this is not surprising because we are using the same system and controller. The only difference is that we have chosen a different control target.

**5. Conclusion.** In this paper we have shown how to spatially discretize an infinite dimensional piezoelectric beam in pH form with a quasi-static electrical field. Moreover, we have shown that the proposed discretization method preserves pH structure of the system, in contrast to the standard discretization methods [20]. The model that we have derived in this paper can now be used to design a finite dimensional controller for a 1-D inflatable structure—one can use, for example, passivity based control methods. In future work we will extend these modeling and discretization techniques to 2-D problems. Additionally in future work we will further extend it to other multiphysic systems with nonconstant interconnection structures, e.g., Navier–Stokes equations.

Additionally in the future we aim at investigating the numerical properties of the proposed method, specifically the convergence, the accuracy of the finite dimensional approximation, and the relation between the finite and infinite dimensional structure.

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